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ON VON KARMAN'S EDDY VISCOSITY IN BOUNDED FLOW

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FOREWORD

This report was prepared by the Fuels and Lubricants Branch, Nonmetallic Materials Laboratory, Directorate of Materials and Processes, Aeronautical Systems Division with Jon Lee as project engineer. The work reported herein was initiated under Project No. 7340, "Nonmetallic and Composite Materials", Task No. 734008, "Power Transmission Heat Transfer Fluids".

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ABSTRACT

For a bounded turbulent flow, the von Karman's eddy viscosity based on the similarity hypothesis yields a well confirmed asymptotic logarithmic velocity distribution for a large Reynolds number. By solving exactly the equation of motion arising from the introduction of the von Karman's eddy viscosity, the notorious sources of inconsistency as (i) the discontinuity of velocity derivative at the matching point, and (ii) the non-vanishing velocity gradient at the enter of a channel, can be removed entirely. Yet the exact solution can be made sufficiently close to the asymptotic logarithmic profile for all other ranges by an arbitrary choice of the parameter separating the conventional two regions in a turbulent flow channel.

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Introduction

For a long time the notorious difficulty of turbulent problems has well been recognized. As an example, heretofore, a rational approach to the description of velocity distribution in a simple flow geometry, such as channel or tube flow, has not yet been realized. The most fruitful and practical endeavour in this direction has been the phenomenological approach for which the analogy of turbulent shear stress to the gradient-type mechanism plays an essential role. Thus, the turbulent flow problems are usually reduced to the finding of plausible expressions for the eddy viscosity which depends on the location in the flow region in contrast to the molecular viscosity.

At present, the eddy viscosities proposed by Deissler[3] and von Karman[10] seem to be more popular than others which have the disadvantage of being less sophiscated or unprofitably complicated.[7] The former predicts the velocity distribution near the wall. It has the virtue of covering the so-called laminar sublayer and buffer layer continuously. The latter, which is valid away from the wall, predicts the well-known logarithmic velocity profile in the turbulent core region. The logarithmic nature of velocity distribution has been validated by experiment and dimensional reasoning, and it is believed that such a function should be an asymptotic profile for a large Reynolds number-thus is called the universal velocity distribution.

It would be an amiss not to mention that, without evoking to the phenomenological approximation, Pai[8] has obtained a semi-empirical polynomial for the velocity distribution which is applicable across the entire channel width. It requires one disposable constant evaluated at each Reynolds number, and the generalization and inherent limitation have recently been studied by Brodkey.[2]

Apart from its usability for a wide range of Reynolds numbers, the principal sources of inconsistency of the forementioned phenomenological approximation are that the two velocity profiles meet abruptly at the intersection and the velocity gradient at the center of a channel does not vanish. These are physically intolerable because the velocity distribution must be not only continuous but also smooth, and the flow pattern must be symmetric and smooth at the center. The direct consequence of non-vanishing velocity gradient at the center is that the maximum velocity (occuring at the center) can not be predicted precisely. (It must be noted that the Pai's method necessitates the knowledge of maximum velocity). Therefore the piece-meal velocity distributions can not provide important information on the ratio of average to maximum velocities.

It has been found that the above two objections can be removed totally by considering the exact equation of motion due to von Karman's eddy viscosity. It is a second order differential, therefore, one can impose only two conditions of continuity and smoothness at the intersection, if one so desires. However, fortunately, the other desideratum of zero-gradient atthe center can automatically be satisfied on account of the singularity at the center being a nodal point.

In retrospect, at high Reynolds number it is clear that the logarithmic profile prevalent in most of the flow region can also be observed by solving the exact differential equation, as one anticipates. Furthermore, the approximation of neglecting the molecular viscosity term prior to solution does not only deprive the possibility of meeting the rational boundary conditions, but also forces one to assume an unnatural infinite velocity gradient at the wall.

Statement of Problem

For a stationary channel flow between two parallel flat plates, the averaged equation of motion takes the following form,

$$(1) \qquad -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

where x is in the direction of mean flow and y is perpendicular to x originating from the wall as shown in Figure 1, and τ_{xy} denotes the total shear stress.

Equation 1 can be expressed in a more obvious form by integrating once in y with the boundary condition, $\tau_{xy} = 0$ at $y=y_c$;

$$(2) \qquad \tau_{xy} = \tau_w \ (1 - y/y_c)$$

For the laminar problem, the shear stress can simply by expressed by the Newton's law of viscosity, i.e., $\tau_{xy} = \mu\, du/dy$, in which μ depends explicitly on the nature of the fluid. However, for turbulent flow, one must take an account of the Reynolds shear stress arising from the temporal average of fluctuating velocity components. Since the exact form of Reynolds (turbulent) shear stress can not be deduced entirely from the theory itself, the only alternative must necessarily be based on a heuristic empiricism. This is undoubtedly the difficulty and weakness of turbulent flow problem, but much success has been made by the phenomenological approach in which the Reynolds shear stress is assumed to be describable by a gradient-type mechanism as

(3) $$\tau_{xy} = (\mu + \tilde{\mu})\, du/dy$$

in which $\tilde{\mu}$ represents the eddy viscosity.

At present the amicably accepted eddy viscosities, to which we shall restrict ourselves here, are;

Deissler's formula near the wall,[3]

(4) $$\tilde{\nu} = \tilde{\mu}/\rho = n^2uy(\,1 - \exp(-n^2uy/\nu))$$,and

von Karman's formula away from the wall,[10]

(5) $$\tilde{\nu} = \tilde{\mu}/\rho = \kappa^2 \left| \frac{(du/dy)^3}{(d^2u/dy^2)^2} \right|$$

where n and κ are disposable constants.

A set of differential equations can be obtained by equating equation 2 and 3, if one further substitutes equations 4 and 5, respectively, for the different ranges. By introducing the usual dimensionless coordinates, $u^+ = u/u^*$ and $y^+ = yu^*/\nu$,where u^* denotes the wall-friction velocity, we finally obtain:

(6) $$\frac{du^+}{dy^+} = \frac{(\,1 - y^+/y_c^+)}{1 + n^2u^+y^+(1 - \exp(-\,n^2u^+y^+))} \qquad \text{for } 0 \le y^+ \le y_L^+$$

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and

$$(7) \qquad \frac{d^2u^+}{dy^{+2}} = \frac{-\kappa\ (du^+/dy^+)^2}{\left[(1 - y^+/y^+_c) - du^+/dy^+\right]^{0.5}} \qquad \text{for } y^+_L \leq y^+$$

where y^+_L separates the ranges of equation 6 and 7, and can be thought of as third constant (e. g., Deissler suggested $y^+_L = 26$). In obtaining equation 7, a priori assumptions that $du^+/dy^+ \geq 0$ and $d^2u^+/dy^{+2} \gtreqless 0$, are tacitly evoked.

It must be observed that the differential equation 6, resulting from the Deissler's contribution, has the merit of replacing the two conventional piece meal velocities for the laminar and buffer layer near the wall. Thus, with this the description of turbulent velocity distribution still contains three empirical constants, the complete elimination of which does not seem so promising within the near future.

Classical Solution

Both equations 6 and 7 are very complicated and non-linear, thus they can not be solved analytically unless some approximations are introduced. Within the proposed range of $y^+ \leq 26$, by approximating $(1 - y^+/y^+_c) \cong 1$, Deissler obtained the solution of equation 8 by numerical iteration,

$$(8) \qquad u^+ = \int_0^{y^+} \frac{dy^+}{1 + n^2u^+y^+(1 - \exp(-n^2u^+y^+))}$$

and presented the solution in a graphical form (8).

For equation 7 which is supposed to be valid at distant from the wall, the effect of molecular viscosity can be considered inferior to the eddy viscosity (1). Following von Karman, we can also assume the constant shear stress near the wall for further simplification (this is indeed contradictory to the previous assumption, but a generally accepted practice as in the case of Prandtl mixing length theory). Had these approximation been introduced in the course of deriving equation 7, one should instead have obtained the following equation,

$$(9) \qquad d^2u^+/dy^{+2} = -\kappa\,(du^+/dy^+)^2 \qquad \text{for } y_L^+ \leq y^+$$

Solution of equation 9 is a familiar logarithmic velocity distribution as;

$$(10) \qquad u^+ - u_L^+ = \frac{1}{\kappa} \ln(y^+/y_L^+)$$

where u_L^+ denotes u^+ at y_L^+.

The actual experimental data seem to confirm the validity of equations 8 and 10, provided the flow is completely turbulent, independent of Reynolds number (3). It might be of some interest in this connexion to note the transitional velocity distributions obtained by Schlinger et al. (9). One virtue of the universal velocity distribution is that it tends to smooth out the irregularities on account of its semi-log representation. On the other hand, the obvious objection and inconsistency are the abrupt change in the curvature of velocity profile at $y_L^+ = 26$ and the non-vanishing velocity gradient at the center of a channel. At this point, we should perhaps be content with the presence of three disposable constants as long as they are unique, and almost independent of Reynolds number. Yet, whatever the artificial division of the flow region might be, we would like to find a continuous velocity distribution (at least, up to first derivative), and to satisfy the physical symmetry condition at the center.

Rational Solution

Equation 6 is a well-behaving regular first order differential equation, even though it does not allow us to obtain a solution in a closed form, and can be solved numerically with the initial condition, $u^+=0$ at $y^+=0$. As the logarithmic profile is an asymptotic solution for a large y_c^+ (or Reynolds number), we can also expect that the exact solution of equation 7 should exhibit such a functional behaviour with the possible exception at the end points.

Let us denote the solution of equations 6 and 7 by u_1^+ and u_2^+, respectively, and also write equation 7 as a corresponding system of the first order differential equations, i.e.;

$$\frac{du_2^+}{dy^+} = P \tag{11}$$

$$\frac{dP}{dy^+} = \frac{-\kappa P^2}{\left[(1 - y^+/y_c^+) - P\right]^{0.5}} \tag{12}$$

We wish to satisfy the following boundary conditions;

(i) $u_2^+ = u_1^+$ at $y^+ = y_L^+$ (continuity)

(ii) $P = du_1^+/dy^+$ at $y^+ = y_L^+$ (smoothness)

(iii) $p = 0$ at $y^+ = y_c^+$ (symmetry)

Usually the system of two first order differential equations can satisfy only two of the above conditions;therefore,the possibility of satisfying the third condition must be examined carefully.

As discussed in the appendix,the end point y_c^+ is a nodal point of the singular differential equation 12,therefore,the family of solutions P with different initial values at y_L^+ must necessarily terminate as P=0 at y_c^+. In other words,the preservation of the molecular viscosity term in equation 12 (represented by -P in the denominator) does allow us to remove the physical inconsistency associated with the classical approximation solution by meeting all three desired boundary conditions. In contrast to the classical solution,the present exact analysis contains y_c^+ as a parameter* (cf.ref.(4,5)),and it can be shown that it is directly related to the Reynolds number by $2 \times y_c^+ \times u_{av}^+$. It would be of some interest to know that the analytical solution of equation 7 in the absence of only the molecular viscosity also contains a lagarithmic function, and the corresponding velocity gradient at the center is proportional to $1/y_c^+$,which vanishes for a large value of y_c^+.

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* The author is indebted to Professor Robert S.Brodkey for pointing the references to his attention.

Results and Discussion

Equation 6 was integrated* with the initial value of $u^+ = 0$ at $y^+ = 0$ and continued until $y^+ = y_L^+$. Then for $y^+ \geq y_L^+$, the set of equations 11 and 12 was integrated with the initial conditions, $u_2^+ = u_1^+$ and $du_2^+/dy^+ = du_1^+/dy^+$ at $y^+ = y_L^+$. The range of y^+ can not be extended to the singular point; therefore, the neighbourhood of y_c^+ was always deleted and later the value of u^+ at y_c^+ was estimated by the numerical integration of P with P=0 at y_c^+ (using the trapezoidal rule).

In order to obtain a solution which can be compared with the existing theory, we need to assign a value to the three parameters, n , κ , and y_L^+. Even though the problem was originally formulated for the channel flow, it has been the usual practice in the past to correlate the tube flow data with the same universal logarithmic velocity distribution. Therefore, in conformity with the spirit of simplification (see, page 161 of ref. (1)), we shall attempt to follow the two familiar velocity distributions as;

Case I

Equation 8 with n = 0.124 $\qquad 0 \leq y^+ \leq 26$

$$u^+ = u_L^+ + \frac{1}{0.36} \ln (y^+/26) \qquad 26 \leq y^+$$

Case II

$$u^+ = y^+ \qquad 0 \leq y^+ \leq 5$$

$$u^+ = -3.05 + \ln y^+ \qquad 5 \leq y^+ \leq 30$$

$$u^+ = 5.5 + \frac{1}{0.4} \ln y^+ \qquad 30 \leq y^+$$

---

* All the numerical integrations of the differential equation was performed by the fourth order Runge-Kutta method with four-digit accuracy.

The velocity distributions corresponding to the above two cases are plotted in Figures 2 and 3 as references. For either case, the value of $n = 0.124$ associated with equation 6 yields a fairly impeccable and conservative profile for small y^+, as has already been found by Deissler[(3)]. For a large y^+, in order to preserve the same slopes in a semi-log. plot, the values of $\kappa = 0.36$ and 0.4 must be used in equation 12 for the respective cases. However, it has been found that somewhat different values for matching the two regions should be adopted; namely, they are $y_L^+ = 15$ and 23 for the cases I and II, respectively. The arbitrary choice of the values for y_L^+ was made so that the exact velocity distribution does not overshoot the asymptotic profile for most of the region.

For each case, several exact velocity distributions corresponding to the different values of the parameter, y_c^+, are compared with the asymptotic profile as shown in Figures 2 and 3. With the exception of $y_c^+ = 100$ (corresponding to Reynolds number of about 2000, for which the theory is not valid), the exact distributions follow the asymptotic profile closely with the desiderata on the velocity profile being faithfully satisfied. The exact distributions deviate considerably from the asymptotic one at the center as has been observed in the actual experiment; however, the variance is well within the spread of experimental data.

The ratio of the maximum to average velocities in a tube can be obtained from the computed velocity distribution, and the results are compared with the smoothed line drawn through the data of Nikuradse, and Stanton and

Pannell (taken from page 149 of ref.(7)),as shown in Figure 4. The velocity ratios for the two cases do not agree very closely either with each other or with the experimentally smoothed curve(average deviation of the experimental data is ± 1 %). The former indicates that the velocity ratio is quite sensitive to the actual shape of the velocity distribution. The latter suggests one to believe that the actual experimental data at large were somewhat different from the asymptotic profiles,provided a sufficient care has been exercised in obtaining the average velocity. Of course,for small Reynolds numbers,that is for Re $<$ 500,the disagreement becomes acute,for which the validity of the present theory must be reviewed.

Conclusions

For a large Reynolds number,the exact solution of the equation of motion based on the von Karman's eddy viscosity approaches the traditional asymptotic logarithmic profile. In addition,the physical inconsistencies manifested by the abrupt velocity change at the matching point and the non-vanishing velocity gradient at the center of a channel were removed in toto. In view of the insignificant discrepency existing between the exact and asymptotic velocity distributions on one hand,and the considerable amount of a computational labour involved in obtaining the exact ones on the other hand,it is fair to say that the findings of the present work may simply be of academic interest for the channel flow. However, a significant error would be committed by using the logarithmic velocity profile,when one considers a flow of thin film over a plate as in the case of turbulent film condensation and two-phase annular flow problems.

APPENDIX 1

Behaviour of equation 12

The imposing of the third boundary condition, i.e., $P = 0$ at $y^+ = y_c^+$, makes equation 12 singular at y_c^+ (6). With no loss of generality, let us now write equation 12 in a more convenient form as,

$$\text{(A-1)} \qquad \frac{dW}{dZ} = \frac{-W^2}{\sqrt{(1-Z)-W}} \qquad (W=0 \text{ at } Z=1)$$

Further by letting $X = 1 - Z$, we can bring the singular point into the origin,

$$\text{(A-2)} \qquad \frac{dW}{dX} = \frac{W^2}{\sqrt{X-W}} \qquad (W=0 \text{ at } X=0)$$

For the sake of convenience, if one introduces a new variable as $\phi = X - W$, we then have,

$$\text{(A-3)} \qquad \frac{d\phi}{dW} = \frac{\sqrt{\phi}}{W^2} - 1 \qquad (\phi=0 \text{ at } W=0)$$

Solution of equation A-3 can not be obtained in a closed form by the elementary functions;* however, a considerable amount of insight for the singular behaviour can be obtained from a simple isocline study.

As usual, equation A-3 should hold for a certain fixed slope, say, K_1:

$$\text{(A-4)} \qquad K_1 = \frac{\sqrt{\phi}}{W^2} - 1$$

* The analytical investigation of the singular behaviour will be presented elsewhere.

Therefore, the isocline equation becomes,

$$\text{(A-5)} \qquad \phi = (K_1 + 1) W^4$$

We are interested only in the first quadrant, $\phi \geq 0$ and $W \geq 0$, and some of the solutions are plotted in Figure 5. The most important conclusion is that the origin is a nodal point into which a family of solutions should converge.

The essence of the nodal behaviour of equation A-3 can further be elucidated by examining an equivalent linear problem, for which an analytical solution is available, as follows;

$$\text{(A-6)} \qquad \frac{d\phi}{dW} = \frac{\phi}{W} - 1 \qquad\qquad (\phi=0 \text{ at } W=0)$$

For this case, the isocline equations are a family of straight lines,

$$\text{(A-7)} \qquad \phi = (K_2 + 1) W$$

and the solutions having the origin as a nodal point can be expressed as,

$$\text{(A-8)} \qquad \phi = W \ln(C/W)$$

where C is a parameter characterizing the family of curves as shown in Figure 6.

Notations

C	integration constant
K_1, K_2	isocline constants
P	du_2^+/dy^+
p	mean pressure
Re	Reynolds number
u	mean flow velocity component
u^*	wall-friction velocity ($\sqrt{\tau_w/\rho}$)
u^+	dimensionless velocity coordinate (u/u^*)
u_1^+, u_2^+	soultion of equations 6 and 7,respectively
u_L^+	u^+ at y_L^+
W	variable defined in equation A-1
X	1 - Z ,see equation A-2
x ,y	coordinate system
y^+	dimensionless distance coordinate (yu^*/ν)
y_c	channel width
y_L^+	parameter separating equations 6 and 7
Z	variable defined in equation A-1
Greeks	
κ	von Karman's constant
μ	molecular viscosity
ν	kinematic viscosity (μ/ρ)
ρ	fluid density
τ_{xy}	total shear stress
ϕ	X - W,see equation A-3

Subscripts

av	average across the channel width
max	maximum
w	wall, i.e., at $y = 0$

overscore

$\sim$	turbulent property

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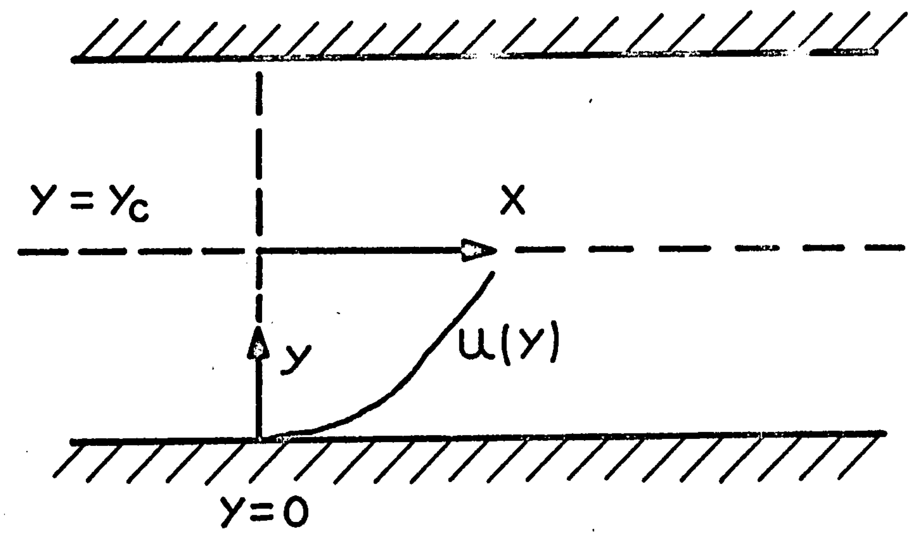

FIGURE I. COORDINATE AXES

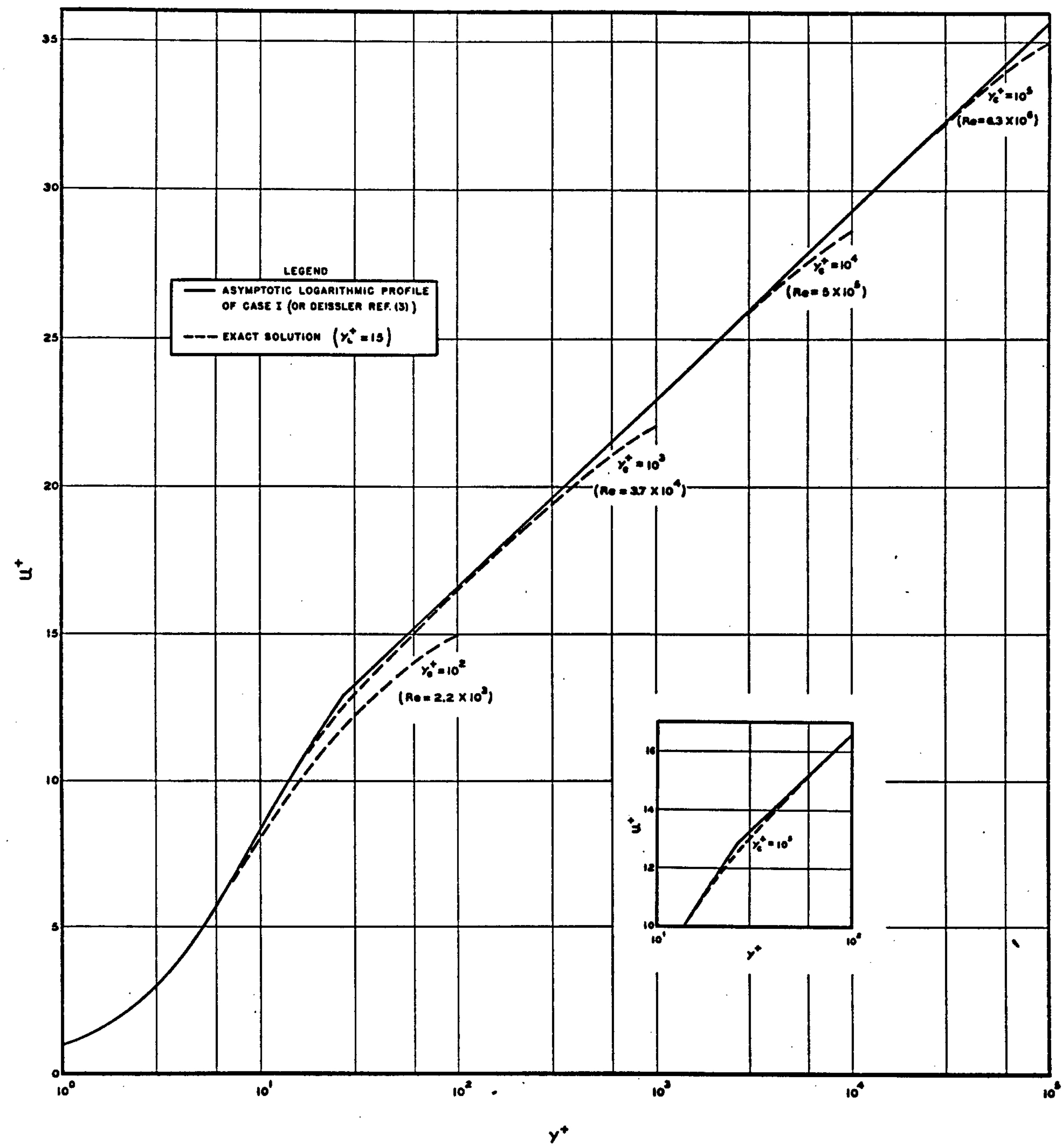

FIG. 2. UNIVERSAL VELOCITY DISTRIBUTION – CASE (I)

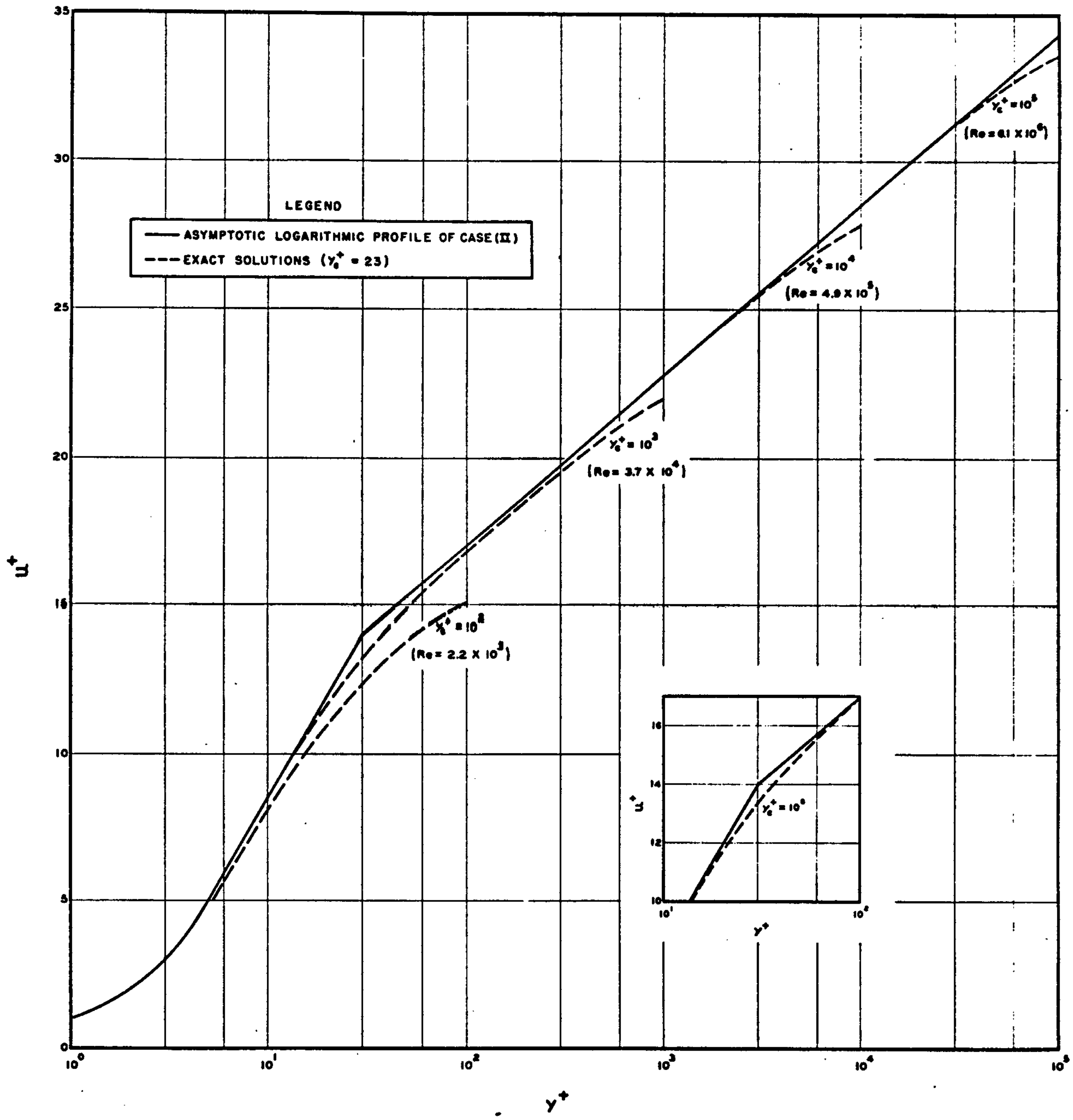

FIGURE 3. UNIVERSAL VELOCITY DISTRIBUTION — CASE (II)

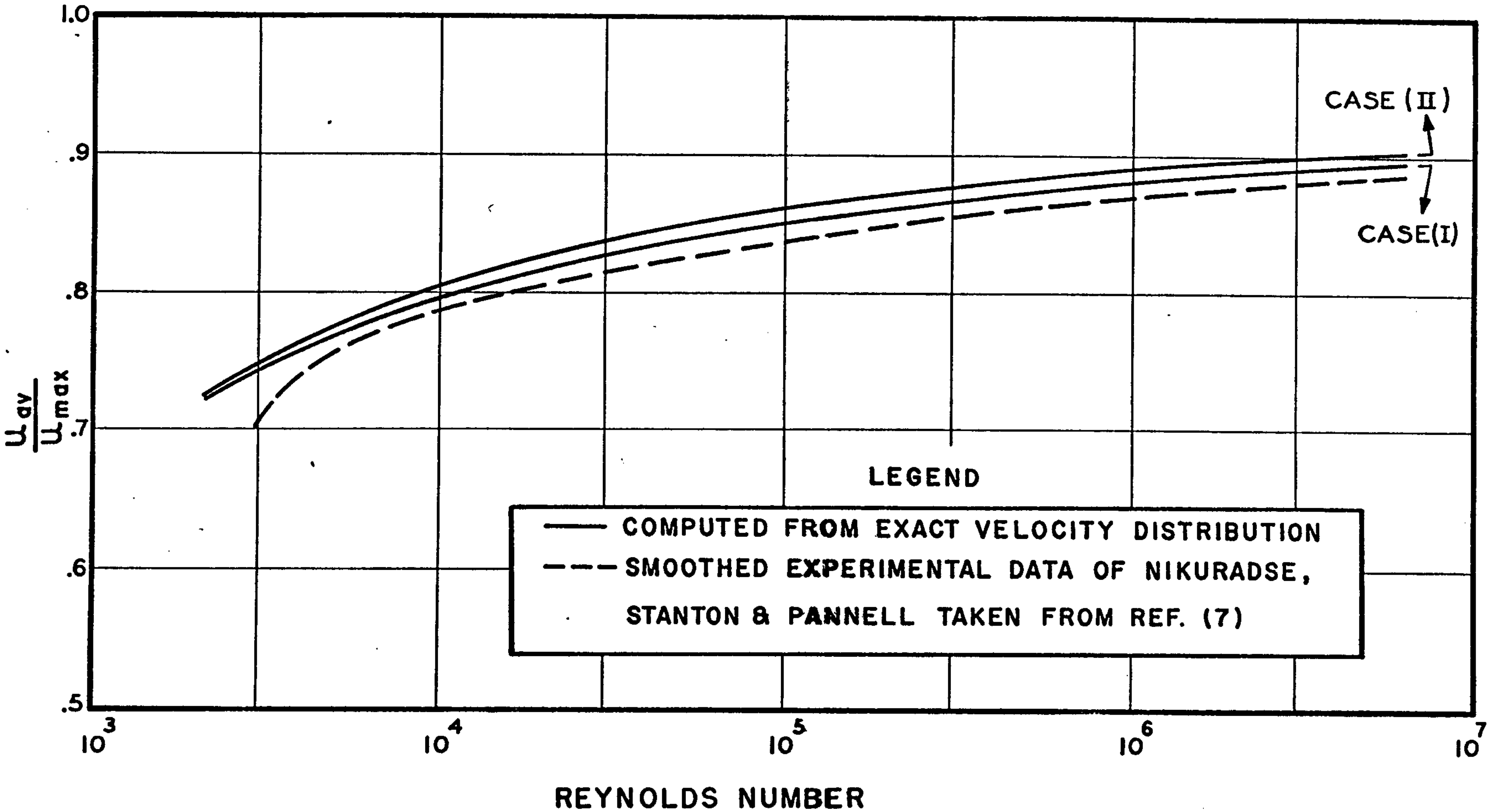

FIGURE 4. VARIATION OF u_{av}/u_{max} VS REYNOLDS NUMBER

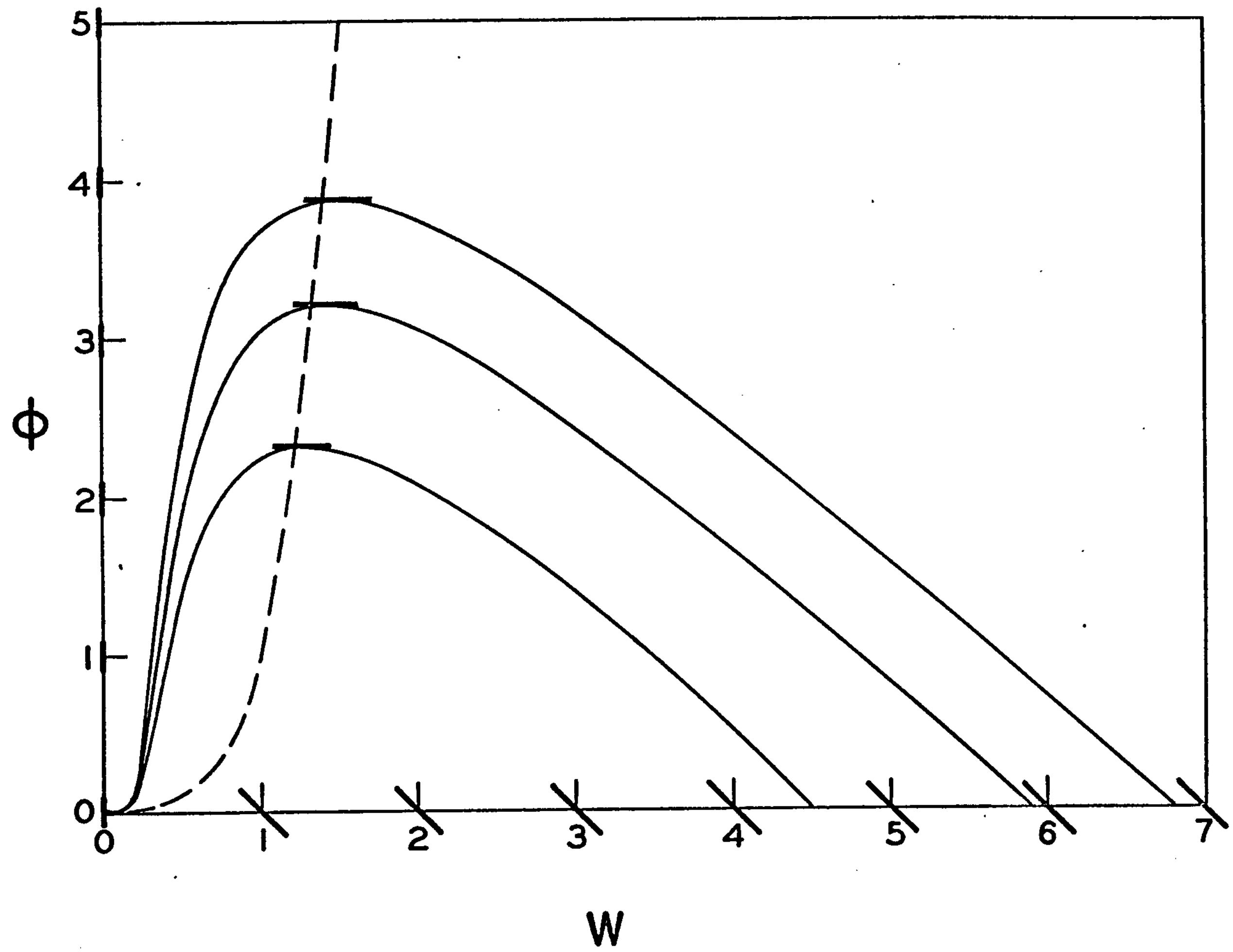

FIGURE 5. ANALOG-COMPUTER PLOT OF EQUATION A-3

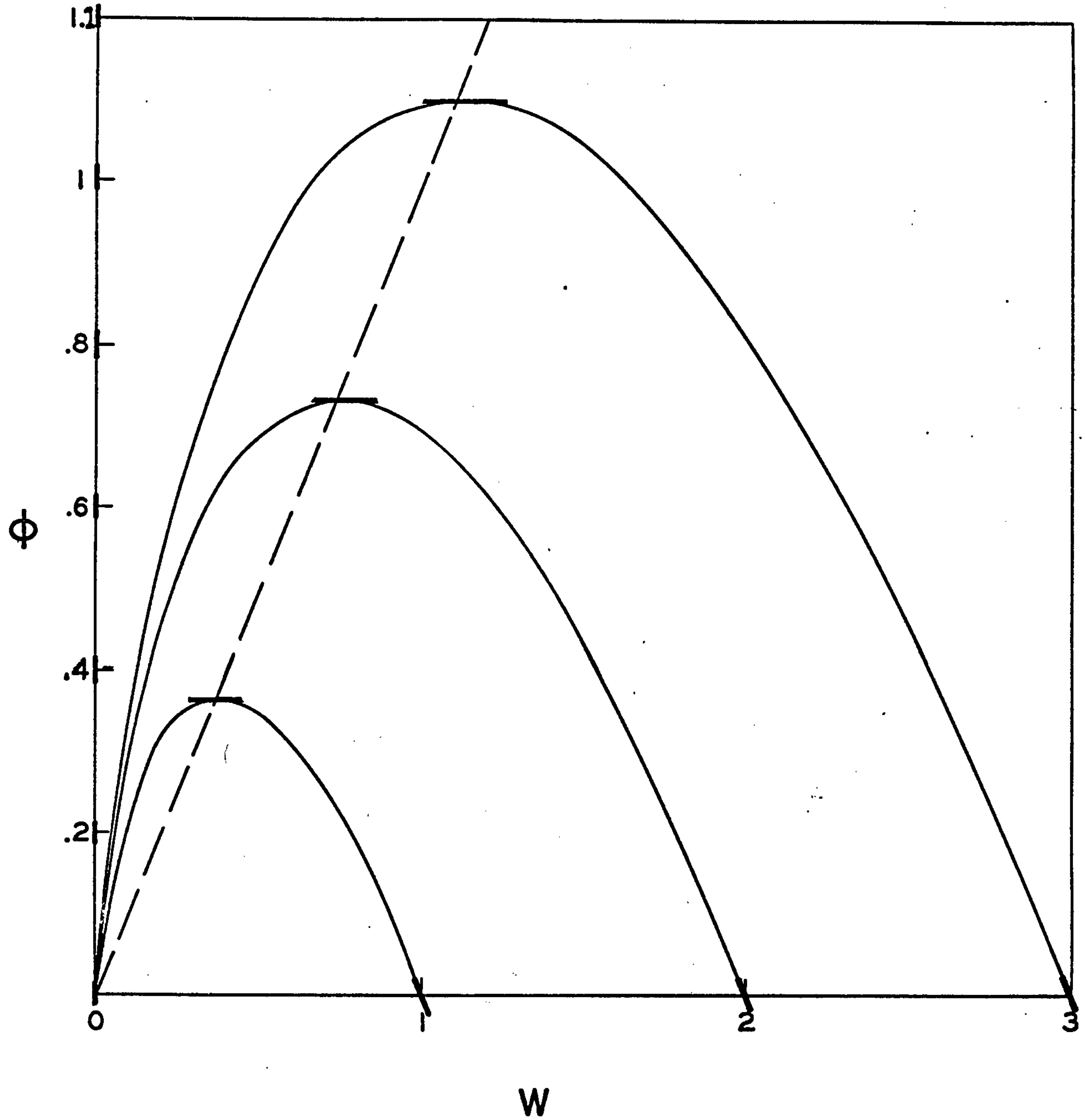

FIGURE 6. PLOT OF EQUATION A-8